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THE CAUCHY PROBLEM FOR A QUASILINEAR SYSTEM WHEN THERE ARE CHARACTERISTIC POINTS ON THE INITIAL SURFACE*

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The problem of the existence, uniqueness and analyticity of a solution of the Cauchy problem in complex and real spaces for a quasilinear analytical set of equations are examined, when the initial data are specified on an analytical surface containing characteristic points, and an error occurs in the initial data and set of equations. In particular, the Cauchy problem with initial data on the envelope of one of the families of the characteristic surfaces of the system is examined.

Discontinuities, whose trajectories are envelopes of the characteristic surfaces, are encountered when studying Chapman-Zhug detonation waves in gas dynamics /1—3/ and magneto-hydrodynamics /4, 5/, and also in the theory of avalanches /6/. The construction of solutions around envelopes of the characteristic surfaces is interesting both in connection with the new problems of detonation in gases - taking into account the inhomogeneity of the background, intakes of mass, momentum and energy to the gas and distortion of the wave front - and in connection with other models.

Investigations of similar problems have so far been confined to linear systems /7—12/, whose knowledge of the order of contact of the characteristic surfaces and initial manifold was substantially used.

1. Consider the set of first-order quasilinear equations in the m-dimensional complex space $x_1, \ldots, x_m$ whose coefficients and right-hand sides are complex functions analytic in the variables $x_1, \ldots, x_m, u_1, \ldots, u_n$

$$\sum_{k=1}^{m}\sum_{j=1}^{n} a_{ijk}\frac{\partial u_j}{\partial x_k} + b_i = 0, \quad i = 1, \ldots, n \; . \tag{1.1}$$

Suppose the analytical initial values of the unknown functions are given on the analytical surface S of complex codimensionality 1, such that the surface S is an envelope of one of the families of the characteristic surfaces of (1.1). We can assume, without loss of generality, that $u_i|_S = 0\ (i = 1, \ldots, n)$ and in some domain D the surface S is specified by the relation $x_1 = 0$. The well-known conditions of non-solvability (1.1) relative to $\partial u_i/\partial x_1\ (i = 1, \ldots, n)$: $\operatorname{rank}\{a_{ij1}\} = n - 1$, $\operatorname{rank}\{a_{ij1} \mid b_i\} = n$ hold on the surface S: $x_1 = 0$. The latter condition can be written in the form

$$\begin{vmatrix} b_1 & a_{121} & a_{131} & \ldots & a_{1n1} \\ b_2 & a_{221} & a_{231} & \ldots & a_{2n1} \\ \vdots & \vdots & \vdots & & \vdots \\ b_n & a_{n21} & a_{n31} & \ldots & a_{nn1} \end{vmatrix} \neq 0 \, . \tag{1.2}$$

Hence it follows that there is no classical solution to the Cauchy problem with initial data on the envelope of characteristic surfaces.

We shall investigate the problem of the existence of the continuous functions $u_i\ (i = 1, \ldots, n)$, which satisfy the initial conditions on S and (1.1) outside S.

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Definition. We shall call the functions u_i $(i = 1, \ldots, n)$ a (generalized) solution of the Cauchy problem for (1.1) in the domain U if they are continuous in $\bar{U}$, take initial values on $\bar{U} \cap S \neq \varnothing$ and satisfy (1.1) everywhere in U, with the exception, possibly, of the points of the analytical set of dimensionality no higher than $m-1$.

Selecting the quantities $u_1, x_2, \ldots, x_m$ as the new independent variables, and the quantities $x_1, u_2, \ldots, u_n$ /2/ as the independent functions, we obtain the non-linear system

$$\sum_{k=2}^{m}\sum_{j=2}^{n}\left(a_{ijk}\frac{\partial u_j}{\partial x_k}+b_i\right)\frac{\partial x_1}{\partial u_1}+\sum_{j=2}^{n}\left(a_{ij1}-\sum_{k=2}^{m}a_{ijk}\frac{\partial x_1}{\partial x_k}\right)\frac{\partial u_j}{\partial u_1}-$$
$$\sum_{k=2}^{m}a_{i1k}\frac{\partial x_1}{\partial x_k}+a_{i11}=0, \quad i=1,\ldots,n. \tag{1.3}$$

In the new variables the initial data $x_1 = 0, u_i = 0$ $(i = 2, \ldots, n)$ are specified on the surface $S: u_1 = 0$. On this surface the determinant of the matrix, composed of the coefficients of the partial derivatives with respect to u_1, is identical with the determinant (1.2), taken for $x_1 = 0$. On the other hand, the determinant of the principal minor of the extended matrix, corresponding to $\partial x_1/\partial u_1$ and calculated on the surface $u_1 = 0$, is identical with the determinant of the matrix $\{a_{ij1}\}$, calculated for $x_1 = 0$. Hence it follows that the Cauchy-Kovalevskii theorem is applicable to the Cauchy problem for (1.3) with homogeneous initial data on the surface $u_1 = 0$ (we shall call this problem conjugate to the initial problem). The theorem guarantees the existence, uniqueness and analyticity of the solution in the neighbourhood of the initial surface, whilst the expansion in series for x_1 begins with a term of an order not less than two

$$F(u_1, x) \equiv x_1 - \sum_{q=p}^{\infty} x_{1q}(x')u_1^q = 0, \quad x' = (x_2, \ldots, x_m), \quad p \geqslant 2 \tag{1.4}$$

$$u_i - \sum_{q=1}^{\infty} v_{iq}(x')u_1^q = 0, \quad i = 2, \ldots, n. \tag{1.5}$$

The quantities $x_{1q} = \partial^q x_1 / \partial u_1^q |_{u_1=0}$ and $u_{iq} = \partial^q u_i / \partial u_1^q |_{u_1=0}$ $(i = 2, \ldots, n)$ are determined in the well-known way /13/ using (1.3), and are analytical functions of both arguments on S. We should obviously take the domain D_1 as the domain of definition of the functions $x_1, u_2, \ldots, u_n$ in the space u_1, x', in order that, on the one hand, series (1.4) and (1.5) converge and, on the other, the point $x_1(u_1, x'), x'$ falls in the domain D.

We shall proceed from the variables u_1, x' to the former variables. Suppose p is the index for the first coefficient, identically non-zero on $S \cap D$ in expansion (1.4). The set of zeros of the function $x_{1p}(x')$ comprises the analytical set $M \subset S$ of complex codimensionality 2 /14/, which, thereby, does not divide S. Consider the point $a = (0, a') \in S \cap D: x_{1p}(a') \neq 0$. According to Weierstrass's preparatory theorem, there is a neighbourhood V_a of the point $(0, 0, a')$ in the space of the variables u_1, x, in which

$$F(u_1, x) = [u_1^p + \alpha_1(x)u_1^{p-1} + \ldots + \alpha_p(x)]\,\Omega(u_1, x) \equiv P_a\Omega.$$

Here α_i $(i = 1, \ldots, p)$ and Ω are function which are analytical in V_a, whilst $\Omega(0, a) \neq 0$ and $\alpha_i(a) = 0$ $(i = 1, \ldots, p)$.

Eq.(1.4) is thus equivalent to $P_a = 0$. The pseudopolynomial P_a has exactly p continuous roots $u_1^{(j)} = u_1^{(j)}(x)$ $(j = 1, \ldots, p)$ analytic in U_a everywhere, with the exception of the points of the discriminant set $\Delta_a \subset U_a$, where the pseudopolynomial P_a has at least one multiple root.

We shall show that $\Delta_a = S \cap U_a$. Suppose x' is an arbitrary point from $S \cap U_a$. Fixing x', we shall apply the Pusy theorem on series inversion /15/ to (1/4)

$$u_1 = \sum_{q=1}^{\infty} u_{1q}(x')x_1^{q/p}. \tag{1.6}$$

When x' is fixed, the p-valued function (1.6) is analytic in some neighbourhood of the point $x_1 = 0$ with the exception of the point $x_1 = 0$ itself, where all p branches coincide. Since at each point of its own domain of convergence the function (1.6) satisfies Eq.(1.4), each branch (1.6) is a root of the pseudopolynomial P_a and, conversely, we shall represent each root in the form (1.6). It follows from this that the series (1.6) converge uniformly inside U_a with respect to x' and that the functions $u_{1q}(x')$ $(q = 1, 2, \ldots)$ are analytic on $S \cap U_a$.

Let us now consider the transition to the former variables in the vicinity of the point $b = (0, b') \in M$, where $x_{1p}(b') = \ldots = x_{1s-1}(b') = 0$, $x_{1s}(b') \neq 0$, $p < s < \infty$. The set defined by these conditions has complex dimensions, no higher than $m-2$, and does not divide S. As before, the neighbourhood V_b of the point $(0, 0, b')$ exists in the space u_1, x, in which (1.4) is equivalent to equating to zero some Weierstrass pseudopolynomial of degree s, whose coefficients are analytic on $U_b = V_b \cap D$

$$P_b \equiv u_1^s + \beta_1(x) u_1^{s-1} + \ldots + \beta_{s-1}(x) u_1 + \beta_s(x) = 0 .$$

In U_b there exist exactly s continuous roots P_b, analytic everywhere on U_b, with the exception of the points of the discriminant set $\Delta_b \subset U_b$, which are the points of the branching of the roots P_b. Obviously, $S \cap U_b \subset \Delta_b$. On the other hand, the point $a \in U_b \cap S \setminus M$, in whose vicinity $U_a \subset U_b$ only p functions $u_1^{(1)}, \ldots, u_1^{(p)}$ exist, satisfying (1.4) can always be found. By virtue of $U_a \subset U_b$ these p functions will be among the roots of the pseudopolynomial P_b. Its remaining $s-p$ roots $u_1^{(p-1)}, \ldots, u_1^{(s)}$ vanish on S only at the points of the analytical set $x_{1p}(x') = \ldots = x_{1s-1}(x') = 0$. The set of roots $u_1^{(1)}, \ldots, u_1^{(s)}$ can be considered to be a multivalued function, analytic on $U_b \setminus \Delta_b$. A continuous transition from one branch to another can be achieved along the curve which passes through the point from Δ_b: for example, the quantities $u_1^{(1)}, \ldots, u_1^{(p)}$ change into one another on $U_b \cap S \subset \Delta_b$.

The analytic set Δ_b is determined by equating to zero the discriminant of the pseudopolynomial P_b which has a zero of the order of $s(s-1)$ at the point b and a zero of the order $p(p-1)$ at the points $S \setminus M$. Consequently, according to Weierstrass's preparatory theorem, S does not exhaust Δ_b and the transition from the branches $u_1^{(1)}, \ldots, u_1^{(p)}$ to the branches $u_1^{(p+1)}, \ldots, u_1^{(s)}$ can also occur outside S.

The set $\Delta_b \setminus S$ is characterized by the fact that $u_1^{(k)}$ $(k = 1, \ldots, s)$ are continuous on it, but do not have a derivative, i.e. they experience a slight discontinuity. As is well-known /13/, slight discontinuities are exclusively allowable along the surfaces have a characteristic direction. Therefore, the set $\Delta_b \setminus S$ consists of the characteristic surfaces which pass through the point b and belong to various characteristic families, although possibly not to them all.

We can sum up all that has been said in the following theorems.

Theorem 1. For any point $a = (0, a') \in S \cap D$: $x_{1p}(a') \neq 0$ a neighbourhood $U_a \subset D$ exists, in which a solution of Cauchy's problem (in the sense defined above) for (1.1) exists and is a p-valued function, analytic on $U_a \setminus S$, and can be represented in the form of series converging on U_a

$$u_i = \sum_{q=1}^{\infty} u_{iq}(x') x_1^{q/p}, \quad i = 1, \ldots, n . \tag{1.7}$$

Here p is the index of the first coefficient which is not identically equal to zero on S in the expansion (1.4).

The statement of Theorem 1 is obtained by substituting (1.6) into Eq.(1.5). In practice, the quantity p is easiest of all to determine by substituting series (1.7) with the undetermined coefficients into (1.1) /2/.

Theorem 2. For the point $b = (0, b') \in S \cap D$, such that $x_{1p}(b') = \ldots = x_{1s-1}(b') = 0$, $x_{1s}(b') \neq 0$ $(p < s < \infty)$, a neighbourhood $U_b \subset D$ exists, in which a solution of Cauchy's problem (in the sense defined above) exists and is an s-valued analytical function, analytic on $U_b \setminus \Delta_b$, where Δ_b is an analytic set which is the discriminant for the pseudopolynomial P_b containing the set $S \cap U_b$ and is not exhausted by them. On $U_b \cap S \setminus M$ only p branches of the solution $u_i^{(1)}, \ldots, u_i^{(p)}$ $(i = 1, \ldots, n)$ assume the initial values, and the remaining $s-p$ branches $u_i^{(p+1)}, \ldots, u_1^{(s)}$ $(i = 1, \ldots, n)$ serve as the first possible continuations at the points $\Delta_b \setminus (S \setminus M)$. The set $\Delta_b \setminus S$ is exhausted by the set of surfaces having a characteristic direction and passing through the point b.

Note that if at some point $b = (0, b')$: $x_{1q}(b') = 0$ $(q = p, p+1, \ldots)$, then by virtue of (1.4) and the theorem of the uniqueness of the analytical functions /14/ $x_1 \equiv 0$ in D. This means that the functions u_i $(i = 1, \ldots, n)$ do not depend on x_1 and we must reduce the number of independent variables in the initial system (1.1).

Corollary. Suppose the surface S is uncharacteristic everywhere, with the exception of the points of its subset M_0. Then: 1) any point $b = (0, b') \in M_0 \cap D$ possesses the neighbourhood U_b, where the existence and uniqueness of the solution is determined by Theorem 2, in which we must put $p = 1$, and s equals the minimum value of q, for which $x_{1q}(b') \neq 0$ in the expansion (1.4); 2) M_0 is an analytic set of dimensions $m-2$.

The proof of 1) does not differ from that of Theorem 2, and the proof of 2) is obvious, since the set M_0 is determined of necessity by the analytical relation $x_{11}(x') = 0$, where

$x_{11}(x')$ is the first coefficient in the corresponding expansion of the solution of the problem, conjugate to the initial problem.

The proven property of the set M_0 is a corollary of the analyticity of the coefficients of (1.1), its right-hand sides, the initial manifold S and the initial data. If $\dim M_0 < m-2$, then a solution of Cauchy's problem does not exist in the neighbourhood $U_b \subset D$ of any point $b \in M_0$. This fact was established earlier for linear systems (/7/, p.27/).

2. The discussion in para.1 enables us to obtain analogies of Theorems 1 and 2 for systems with real-valued analytical coefficients, right-hand sides and initial data in the real space R^m. A domain $D \subset R^m$ exists, in which, after an appropriate substitution, the initial surface S is given by the relation $x_1 = 0$ and the initial values of the unknown functions are similar. As in para.1, we shall proceed to Cauchy's conjugate problem for (1.3). According to the Cauchy-Kovalevski theorem, its solutions take the form (1.4) and (1.5), where x_{1q} and u_{iq} $(i = 2, \ldots, n)$ are real-valued functions from u_1, x', analytic on $S \cap D_1$ (D_1 is determined as in para.1). The dimensions of the set of M zeros of the analytical function $x_{1p}(x')$ do not exceed $m-2$, but can also be smaller.

The basis of the statements obtained in para.1 is Weierstrass's preparatory theorem and Pusy's theorem on series inversion of the type (1.4). To obtain a real analogy of Weierstrass's preparatory theorem, it is sufficient to consider also the equation - proved in Weierstrass's preparatory theorem - complex-conjugate to it. The reduction of Pusy's theorem to the case of real variables is obvious and is based on separating the real values of the function of the root from the real number. Since the reasoning used for Theorems 1 and 2 will also be useful with insignificant changes when there are real variables, Theorems 1 and 2 will also be useful with insignificant changes when there are real variables, Theorems 1 and 2 will only be prefaced by remaks about the specific features of the case considered.

1°. In the neighbourhood U_b of the point $b = (0, b') : x_{1p}(b') = \ldots = x_{1s-1}(b') = 0$, $x_{1s}(b') \neq 0$ $(s > p \geq 1)$, Eq.(1.4) is equivalent to equating to zero some Weierstrass polynomial P_b with real analytical coefficients which have degree s. In any neighbourhood U_b - as small as desired - of the point b, the point $a = (0, a') : x_{1p}(a') \neq 0$ is obtained, in whose neighbourhood $U_a \subset U_b$, P_b has either one real root, which vanishes on S $(p = 1, 3, 5, \ldots)$, or, from one side of S, two roots, and from the other, no root $(p = 2, 4, \ldots)$. In the latter case it may turn out that points are obtained which are as near b on S as desired, in the region of which a solution exists from various sides of S, i.e. b lies on a surface of the dimensionality $m-2$, which separates S into domains where $x_{1p}(x')$ has different signs.

2°. A change in the number of real roots of the pseudopolynomial P_b can occur - by virtue of the theorems on analytical continuation in /14/ - only on passing through the surfaces of dimensionality $m-1$, consisting of the points Δ_b, and moreover only by an even amount.

3°. As in para.1, the discriminant set $\Delta_a, a = (0, a') : x_{1p}(a') \neq 0$ agrees in the corresponding neighbourhood U_a with the set $S \cap U_a$; the discriminant set Δ_b of the pseudopolynomial P_b can contain, besides surfaces of dimensionality $m-1$, surfaces of lesser dimensionality, in particular points which are isolated relative to Δ_b.

4°. The surfaces of dimensionality $m-1$ from Δ_b divide U_b into a finite set of connected domains U_{bi} $(i = 1, \ldots, k)$. In each such domain the number of real roots P_b is constant everywhere, with the exception, possibly, of the points from Δ_b, the set of which does not divide the domain considered. Among the real roots P_b, however, there may be some which do not take zero values on S anywhere apart from some subset of the set $M = \{x' : x_{1p}(x') = 0\}$.

Theorem 3. Suppose p is the index of the first coefficient $x_{1p}(x')$, not identically equal to zero on $S \cap D$, in the expansion (1.4) of the solution of the problem which is conjugate to the initial problem. Then for each odd p, a neighbourhood U_a exists for each point $a = (0, a') \in S : x_{1p}(a') \neq 0$, in which the solution (in the previously defined sense) of Cauchy's problem for (1.1) exists, is unique, is analytic on $U_a \setminus S$, and can be represented in the form of series converging in U_a with coefficients which are analytic on S

$$u_i = \sum_{q=1}^{\infty} u_{iq}(x') x_1^{q/p}, \quad i = 1, \ldots, n. \tag{2.1}$$

For even p for each point $a = (0, a') \in S : x_{1p}(a') \neq 0$ a neighbourhood U_a exists, which can be divided by the surface S into two parts $U_a^+ : x_1 x_{1p} > 0$ and $U_a^- : x_1/x_{1p} < 0$, possessing the properties: a) in U_a^- a solution of Cauchy's problem for (1.1) does not exists; b) in U_a^+ a solution exists, is a two-valued function which is analytic in U_a^+ and can be represented by series converging in U_a^+ with analytical coefficients on S

$$u_i = \sum_{q=1}^{\infty} u_{iq}(x') (\pm \sqrt[p]{|x_1|})^q, \quad i = 1, \ldots, n. \tag{2.2}$$

Theorem 4. Suppose U_b is the neighbourhood of the point $b = (0, b') \in S : x_{1p}(b') = \ldots =$

$x_{1s-1}(b') = 0,\ \ x_{1s}(b') \neq 0\ \ (s > p \geqslant 1)$, in which Eq.(1.4) is equivalent to the equation $P_b = 0$. The neighbourhood U_b is divided by surfaces of dimensionality $m - 1$, comprising the points of the discriminant set Δ_b, into a finite number of connected domains $U_{bi}, b \in \partial U_{bi}\ (i = 1, \ldots, k)$. In the domains $U_{bi}\ (i = 1, \ldots, r \leqslant k): \partial U_{bi} \cap (S \setminus M) \neq \varnothing$, the problems of the existence, number and analyticity of the solutions of Cauchy's problem are determined by Theorem 3. In the remaining domains $U_{bi}\ (i = r + 1, \ldots, k)$, if they exist, a solution may exist or not exist, but if it exists it is analytical outside the points Δ_b and is not more than s-valued. In the union of the domains U_{bi}, where a solution exists, it does not have a uniform asymptotic form as $x \to b$. If (1.1) is hyperbolic outside S, then the set Δ_b is exhausted by the surfaces which have a characteristic direction.

Note that the answers to the questions of which domains U_{bi} have a solution and which do not - and whether we can also exclude solutions which are continuous in the sum of certain adjacent domains U_{bi}, comprising, for example, one of the two halves into which the surface S divides the neighbourhood U_b - require the formulation of the problem to be given a specific definition. In the literature, classes of Cauchy problems are distinguished for linear sets when $p = 1$ and b is an isolated point of the set M, when the solution is both inhomogeneous /11, 12/, and homogeneous /9–11/ in one of the two halves into which the surface S divides the neighbourhood U_b.

The quasilinearity of the problems considered makes it possible to use the results of this paper in the theory of detonation waves. Consider, for example, the case of a general situation when $x_{12}(x') \neq 0$ in (1.4) (i.e. $p = 2$). The possibility of constructing a flow from it is a necessary condition for the Chapman-Zhug wave to exist. If, from the condition attaching to the Chapman-Zhug wave the flow must be arranged for $x_1 > 0$, then it can be constructed if $x_{12}(x') > 0$, where $x_{12}(x')$ is determined by the conditions on the wave and the model equations (1.1).

The presence of right-hand sides in (1.1) enables us to obtain more general conditions for the existence of Chapman-Zhug waves when there are inflows of mass, momentum and energy in the flow behind a free, arbitrary deformation of the wave front and a background inhomogeneity in front of it. When the Chapman-Zhug wave exists, the parameters of the flow behind the free wave will divide into converging series of the form (2.1) or (2.2). In particular, the known expansions of the solutions as divergent, curvilinear - including cylindrical and spherical - Chapman-Zhug detonation waves, propagating in an inhomogeneous static gas /3, 16/, are converging series.

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